Answer the following:

1- Rewrite the function $f(z) = \frac{\sin z^2}{z^2}$ to be analytic at $z = 0$.

2- Obtain the value of $\oint_c \frac{e^z}{z^2-1}\,dz$, where $c: |z| = 2$.

3- Find the analytic function $f(z) = u + iv$, if $v = r^2 \cos\theta$. Prove that u is harmonic.

4- Find all possible Laurent series for the function $f(z) = \frac{1}{1-z^2}$ with centre at $z_0 = 1$.

5- Evaluate the definite real integral $\int_{-\infty}^{\infty} \frac{2x+1}{(x^2+4)(x^2+9)}\,dx$.

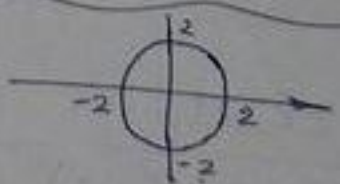
---

1) $F(z) = \frac{\sin z^2}{z^2} \Rightarrow \lim_{z\to 0} \frac{\sin z^2}{z^2} = \frac{0}{0}$ (indeterminate)

$\lim_{z\to 0} \frac{2z\cos z^2}{2z} = \lim_{z\to 0} \cos z^2 = 1$

$\therefore F(z) = \begin{cases} \frac{\sin z^2}{z^2} & z \neq 0 \\ 1 & z = 0 \end{cases}$

2) $\oint_c \frac{e^z}{z^2-1}$ $|z| = 2$



$F(z) = \frac{e^z}{z^2-1} = \frac{e^z}{(z-1)(z+1)}$

$\therefore$ Poles $z = 1, -1$ inside Circle

$\therefore \text{Res}[f(z), -1] = \lim_{z\to -1} \frac{e^z}{(z-1)} = \frac{-1}{2e}$

$\text{Res}[f(z), 1] = \lim_{z\to 1} \frac{e^z}{z+1} = \frac{e}{2}$

$\Rightarrow I = 2\pi i \left[\frac{-1}{2e} + \frac{e}{2}\right]$

$I = 2.35\pi i$

(3) $F(z) = u + iv$, $v = r^2 \cos\theta$

$\therefore v_r = 2r\cos\theta$, $v_\theta = -r^2 \sin\theta$ $\Rightarrow F'(z) = e^{-i\theta}[u_r + iv_r]$

$\because F(z)$ is analytic $\Rightarrow u_r = \frac{1}{r} v_\theta$, $u_\theta = -r v_r$

$\therefore F'(z) = e^{-i\theta}\left[\frac{-r^2}{r}\sin\theta + i 2r\cos\theta\right]$

$F'(z) = e^{-i\theta}\left[-r\sin\theta + i 2r\cos\theta\right]$

For $\theta = 0$, $r = z$

$\therefore F'(z) = -z\sin 0 + i 2z\cos 0$

$\therefore F'(z) = i2z \Rightarrow F(z) = iz^2 + C$

$u_r = \frac{1}{r} v_\theta = -\frac{r^2}{r}\sin\theta = -r\sin\theta$

$u_\theta = -2r^2\cos\theta + k'(\theta)$

$v_r = 2r\cos\theta \Rightarrow$

$\therefore$

$u_{rr} = -\sin\theta$

$u_{\theta\theta} = -2r^2\sin\theta$

$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$

$= -\sin\theta - \sin\theta + 2\sin\theta$

$= 0$

$\Rightarrow u$ is harmonic

(4) $F(z) = \frac{1}{1-z^2}$, $z_0 = 1$

(2)

$\frac{1}{1-(z-1+1)^2} = \frac{1}{1-[(z-1)^2 + 1 + 2(z-1)]}$

$= \frac{1}{1-(z-1)^2 - 1 - 2(z-1)} = \frac{1}{-(z-1)^2 - 2(z-1)}$

$= \frac{1}{-(z-1)[(z-1)+2]} = \frac{-1}{(z-1)}\left[\frac{1}{2+(z-1)}\right]$

$$\frac{-1}{(z-1)}\left[\frac{1}{2}\cdot\frac{1}{\left(1+\frac{z-1}{2}\right)}\right]$$

$$F(z)=\frac{-1}{2(z-1)}\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+-\right]$$

$$=\frac{-1}{2(z-1)}+\frac{1}{4}-\frac{z-1}{2\times 2^2}-\frac{(z-1)^2}{2\times 2^3}+$$

$$\left|\frac{z-1}{2}\right|<1$$

For

$$|z-1|<2$$

$$\frac{1}{z-1}\left[\frac{1}{(z-1)\left(1+\frac{2}{z-1}\right)}\right]$$

$$=\frac{-1}{(z-1)^2}\left[1-\frac{2}{z-1}+\left(\frac{2}{z-1}\right)^2+\right]$$

$$F(z)=\frac{-1}{(z-1)^2}+\frac{2}{(z-1)^3}-\frac{2^2}{(z-1)^4}+\cdots$$

For

$$\left|\frac{2}{z-1}\right|<1$$

For $|z-1|>2$

5)

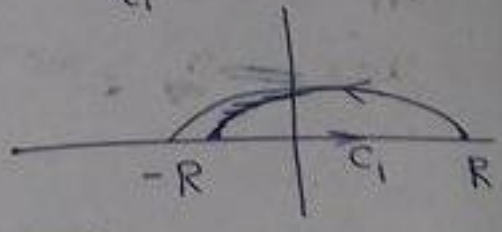
$$\int_{-\infty}^{\infty}\frac{2x+1}{(x^2+4)(x^2+9)}dx=\int_{-\infty}^{\infty}F(x)\,dx$$

$$\oint_C\frac{2z+1}{(z^2+4)(z^2+9)}dz=\int_{C_R}F(z)\,dz+\int_{C_1}F(x)\,dx$$

$$z^2+4=0$$

$$(z+2i)(z-2i)=0$$

$$z_1=-2i\ ,\ z_2=2i$$



$$\text{Res}\left[F(z),z_2(2i)\right]=\lim_{z\to 2i}\frac{(z-2i)}{(z-2i)(z+2i)(z^2+9)}=\frac{-i}{10}$$

$$\text{Res}\left[F(z),z_4(3i)\right]=\lim_{z\to 3i}\frac{(z-3i)}{(z^2+4)(z-3i)(z+3i)}=\frac{i}{30}$$

$$\oint\frac{2z+1}{(z^2+4)(z^2+9)}dz=2\pi i\left[\frac{1}{10i}-\frac{1}{30i}\right)$$

$$=\boxed{\frac{2}{15}\pi}$$

$$\left| \int_{C_R} \frac{2Re^{i\theta}+1}{(R^2e^{2i\theta}+4)(R^2e^{2i\theta}+9)}\,d\theta \right| \le \int_0^{2\pi} \frac{2R+1}{(R^2-4)(R^2-9)}\,|d\theta|$$

$$= \frac{(2R+1)\,2\pi}{(R^2-4)(R^2-9)}$$

$R \to \infty \Rightarrow \int_{C_R} = 0$

$$\therefore \int_{-\infty}^{\infty} \frac{2x+1}{(x^2+4)(x^2+9)}\,dx = \boxed{\frac{2}{15}\pi}$$

$$\oint_{C_R} \frac{2z+1}{(z^2+4)(z^2+9)}\,dz \quad \text{put } z = Re^{i\theta} \Rightarrow dz = Ri\,e^{i\theta}$$

$$\left| \int_{C_R} \frac{(2Re^{i\theta}+1)\,Ri\,e^{i\theta}}{(R^2e^{2i\theta}+4)(R^2e^{2i\theta}+9)}\,d\theta \right| \le$$

$$\int_0^{2\pi} \frac{2R+1}{(R^2-4)(R^2-9)}\,|d\theta|$$

$$= \frac{(2R+1)\,2\pi}{(R^2-4)(R^2-9)} \qquad R \to \infty \Rightarrow \int_{C_R} = 0$$

$$\int_{-\infty}^{\infty} \frac{2x+1}{(x^2+4)(x^2+9)}\,dx = \boxed{\frac{2}{15}\pi}$$

④